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# Money Supply and Capital Accumulation on the Transition Path

Revisited<sup>1</sup>

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## **Abstract**

Fischer (1979) and Asako (1983) analyze the sign of the correlation between the growth rate of money and the rate of capital accumulation on the transition path. Both plug a CRRA utility (based on a Cobb-Douglas and a Leontief function, respectively) into Sidrauski's model – yet return contrasting results. The present analysis, by using a more general CES utility, presents both of those settings and conclusions as limiting cases, and generates economic figures more consistent with reality (for instance, the interest-rate elasticity of the money demands derived from those previous works is necessarily 1 and 0, respectively).

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# 1 Introduction

In a classical paper in the literature, Fischer (1979) has derived two important conclusions:

- (i) although in the Sidrauski model money is superneutral (the steady-state capital stock is invariant to the growth rate of money supply), money is in general not superneutral on transition paths;
- (ii) "for the constant relative risk aversion (CRRA) family of utility functions (except logarithmic), the rate of capital accumulation is faster the higher the growth rate of money".

Fischer's second conclusion has been particularly quoted in the literature because it generates a positive relation between inflation and the rate of capital accumulation, a somewhat unexpected result. However, it is specific of a particular type of CRRA preferences in which the elasticity of substitution between money and consumption equals one (a Cobb-Douglas case).<sup>1</sup> Since such preferences necessarily lead to money demands with a unitary interest-rate elasticity, and there is no particular theoretical reason why this should be so, Fischer's conclusion is certainly non-generic.

Asako (1983) addressed the same issue by considering another type of CRRA preferences. Instead of fixing the elasticity of substitution at 1, he fixed it at 0. In his analysis, real per capita consumption  $c$  and real per capita monetary balances  $m$  are assumed to be perfect complements. Asako's main conclusion is that, in this case, Fischer's second conclusion does not necessarily hold. However, once again we are led to a non-generic conclusion, since in Asako's framework the elasticity of real balances with respect to the nominal interest rate is zero.

This paper extends those two contributions by assuming preferences defined so that the elasticity of substitution between  $c$  and  $m$  can assume any other value in the positive real line, not only one and zero. Instead of preferences based on a Cobb-Douglas or a Leontief utility function, preferences are based here on a CES utility function, which admits those two as limiting cases. The generality regarding the elasticity of substitution extends nicely to the elasticity of real balances with respect

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<sup>1</sup>Throughout this work, by "elasticity" it is meant its absolute value.

to the interest rate. Instead of necessarily being equal to 1 or 0, here it is allowed to take on any positive real value.

Our main task is to study how the sign of the relationship between the growth rate of money supply and the rate of capital accumulation depends upon the model's parameters. Special emphasis is put on the roles of the elasticity of substitution  $\alpha$  (which, incidentally, turns out to be also the interest-rate elasticity of money demand) and  $\sigma$ , a parameter intimately related to the coefficient of relative risk aversion  $\sigma_R$ .

A final word must be given about our revisiting of Fischer's and Asako's results on how capital accumulation may react to money supply. Despite the existing controversy on the role of money supply vis-à-vis the use of short term interest rates in the conduct of monetary policy (see, e.g., Woodford 2008, McCallum 2008 and Nelson 2008), it is not our purpose to add to that discussion here. We find it of interest on its own to note how Fischer's and Asako's apparently contrasting results can be understood as two very particular cases of a more encompassing analysis which leads to a whole new set of results. However, since the model we use allows for the endogenization of the nominal interest rate, as well as the determination of its equilibrium path, those readers who want to interpret our results from the interest-rate-policy perspective are offered a path to proceed in that direction as well.

## 2 Theoretical Setting

The basic setting here is a perfect-foresight version of Sidrauski (1967), as in Fischer (1979) and Asako (1983). A representative agent maximizes  $\int_0^{+\infty} e^{-\delta t} u(c, m) dt$  with respect to  $c, m, k \geq 0$  satisfying the budget constraint  $\dot{k} + nk + \dot{m} + (\pi + n)m = f(k) + x - c$ , where  $c$  is per capita real consumption,  $m$  is per capita real cash balances,  $k$  is per capita capital stock,  $x$  is a lump-sum transfer from the government,  $\pi$  is the (both expected and realized) inflation rate,  $n$  is the

population growth rate,  $m_0, k_0 > 0$  are given, and the time subscripts for every variable are being omitted. The utility function  $u$  is concave and such that  $u_1, u_2 > 0$ ,  $u_{11}, u_{22} < 0$ ,  $J_1 := (u_2/u_1)_1 > 0$  and  $J_2 := (u_2/u_1)_2 < 0$ .<sup>2</sup> The production function  $f$  is such that  $f' > 0$ ,  $f'' < 0$  and the Inada conditions are satisfied.

Assuming an interior solution, the Euler equations give

$$u_1 (f' (k) + \pi) = u_2, \quad (1)$$

$$u_1 (\pi + \delta + n) - u_2 = u_{11} \dot{c} + u_{12} \dot{m}. \quad (2)$$

As in Fischer (1979), government is assumed to issue money at rate  $\theta$ , implying

$$\frac{\dot{m}}{m} = \theta - \pi - n, \quad (3)$$

and run a balanced budget, so that  $x = \dot{m} + (\pi + n)m (= \theta m)$ . In equilibrium, the budget constraint becomes

$$\dot{k} + nk = f(k) - c. \quad (4)$$

Equations (2), (3) and (4) form a system of three first-order differential equations in  $(c, m, k)$ .

Call  $(c^*, m^*, k^*)$  its steady state. Equations (2) and (1) immediately yield  $k^*$ , then (4) can be used to give  $c^*$ , while (2) and (3) give  $m^*$ :

$$f' (k^*) = \delta + n, \quad (5)$$

$$c^* = f(k^*) - nk^*, \quad (6)$$

$$\frac{u_2 (c^*, m^*)}{u_1 (c^*, m^*)} = \theta + \delta. \quad (7)$$

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<sup>2</sup> $(u_2/u_1)_i$  stands for the partial derivative of  $u_2/u_1$  with respect to its  $i$ th argument.

Equations (5) and (6) bring the important conclusion that  $k^*$  and  $c^*$  are superneutral (i.e., independent from  $\theta$ ).<sup>3</sup> The amount  $\theta + \delta$  in (7) should be recognized as the long-term nominal interest rate  $r$  (accordingly,  $\theta + \delta - \pi = \delta + n = f'(k^*)$  is the long-term real interest rate). Since  $u_1, u_2 > 0$ , it is positive, and since  $J_2 < 0$ , it is negatively correlated to  $m^*$  (money is a normal good).

Linearizing the  $3 \times 3$  differential system around  $(c^*, m^*, k^*)$  yields (this is equation 9 in Fischer)

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} mJ_1 \frac{u_{12}}{u_{11}} & mJ_2 \frac{u_{12}}{u_{11}} & -f'' \frac{u_1 + mu_{12}}{u_{11}} \\ -mJ_1 & -mJ_2 & mf'' \\ -1 & 0 & \delta \end{bmatrix} \bigg|_{c^*, m^*, k^*} \begin{bmatrix} c - c^* \\ m - m^* \\ k - k^* \end{bmatrix}. \quad (8)$$

Call  $D$  the preceding  $3 \times 3$  matrix.  $D$ 's characteristic polynomial (see Appendix A) equals

$$(\lambda^2 - \delta\lambda) \left( -\lambda + m \left( J_1 \frac{u_{12}}{u_{11}} - J_2 \right) \right) + f'' \left( \frac{u_1 + mu_{12}}{u_{11}} \lambda + mJ_2 \frac{u_1}{u_{11}} \right), \quad (9)$$

with everything evaluated at the steady state. Since  $\det D < 0$  and  $\text{tr } D > 0$  (see Fischer 1979, p. 1436),  $D$  has one negative eigenvalue (of multiplicity 1) and two eigenvalues with positive real parts (distinct or not).

Equation (7), if explicitly solvable for  $m^*$  in terms of  $c^*$  and  $r$ , can be used to eliminate all the  $m^*$  terms in  $D$  (both the ones explicit there and the ones appearing in the expressions for the derivatives of  $u$  and the  $J_i$ 's), so that all of  $D$ 's dependence on  $\theta$  occurs exclusively because of the  $r$  terms (since  $r := \theta + \delta$ , and  $c^*$  and  $k^*$  are superneutral). In order to focus attention on the effects of changing the growth rate of money supply, call  $D$ 's unique negative eigenvalue  $\lambda(\theta)$ .

As noted in Fischer (1979, p. 1438), if one is sufficiently close to the steady state, the accumulation of capital on the transition path is faster the larger is  $-\lambda(\theta)$ . But in order to know

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<sup>3</sup>In order to be assured that equations (5), (6) and (7) completely determine the steady state, the Inada conditions and  $f'' < 0$  should be used for  $k^*$ , while  $J_2 < 0$  for  $m^*$ .



whether "the larger is  $-\lambda(\theta)$ " translates as "the faster the money supply increases" or "the slower the money supply increases", the sign of  $\lambda'(\theta)$  must be studied. If  $\lambda'(\theta) < 0$ , the former is true, and if  $\lambda'(\theta) > 0$ , the latter. We say that there is "transition-path superneutrality" when  $\lambda'(\theta) = 0$ .

For  $\sigma \in \mathbb{R}_{++}$ , define  $g_\sigma : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$  by

$$g_\sigma(v) = \begin{cases} \frac{v^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log v & \text{if } \sigma = 1 \end{cases}. \quad (10)$$

Using a utility specification equivalent (that is, equal up to an increasing affine transformation) to

$$u(c, m) = g_\sigma \left( c^\rho (\gamma m)^{1-\rho} \right), \quad (11)$$

where  $\rho \in (0, 1)$  and  $\gamma \in \mathbb{R}_{++}$ , Fischer (1979) concluded that, unless  $\sigma = 1$  (in which case  $\lambda'(\theta) = 0$ ), one should expect to have  $\lambda'(\theta) < 0$  (in words, positive correlation between the growth rate of money and the rate of capital accumulation on the transition path).<sup>4</sup> However, using a utility function equivalent to<sup>5</sup>

$$u(c, m) = g_\sigma(\min(c, \gamma m)), \quad (12)$$

with the same restrictions over parameters as above, Asako (1983) showed that in addition to Fischer's results for  $\sigma \leq 1$ , one has  $\lambda'(\theta) > 0$  when  $\sigma > 1$ .<sup>6</sup>

On the one hand, taking a monotonic transformation of Leontief's fixed-proportions function for the utility function (as in Asako 1983) can be considered extreme because it generates a "money-

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<sup>4</sup>This equivalence between (11) and expression 12 in Fischer (1979) can be established by taking  $\sigma := \alpha / (\alpha + \beta)$  and  $\sigma = 1 - (\alpha + \beta)(1 - R)$ , and noting that the restrictions imposed on  $\alpha$ ,  $\beta$  and  $R$  in Fischer ( $\alpha, \beta, R \in \mathbb{R}_{++}$ ,  $\alpha + \beta \leq 1$  and  $R \neq 1$ ) correspond to ours.

<sup>5</sup>Lucas focuses on the case of a homogeneous utility function (i.e.,  $\sigma \neq 1$ ), but the present work does not depend on that assumption.

<sup>6</sup>The correspondence between Asako's terminology and ours can be established by maintaining the same meaning for  $\gamma$ , taking  $\sigma = 1 - \alpha(1 - R)$ , and noting that the restrictions imposed there ( $\alpha > 0$  and  $R > 1 - 1/\alpha$ ) correspond to the restrictions on our parameters.

demand function" in which the interest-rate elasticity of money demand is zero:  $m^* = (1/\gamma) c^* = (1/\gamma) c^* r^0$ . On the other, a monotonic transformation of a Cobb-Douglas function (as in Fischer 1979) is also quite restrictive, since (7) then gives  $m^* = ((1 - \rho) / \rho) c^* r^{-1}$ , a money demand with elasticity necessarily equal to one. Hence we look for a utility function which still rationalizes log-log money-demand functions, but with any given elasticity  $\alpha > 0$ .

The matter is one of integrability, and is answered in Cysne and Turchick (2009) in the following way. Consider Lucas's (2000, section 3) utility function, equivalent to

$$u(c, m) = g_\sigma \left( c \varphi \left( \frac{m}{c} \right) \right), \quad (13)$$

where  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is such that  $\varphi' > 0$  and  $\varphi'' < 0$ .<sup>7</sup> According to the math carried out in example 1 in Cysne and Turchick (2009), in order to generate a money demand

$$m^* = K c^* r^{-\alpha}, \quad (14)$$

with  $K \in \mathbb{R}_{++}$  and  $\alpha \in \mathbb{R}_+ \setminus \{0, 1\}$ , it is necessary and sufficient to plug into (13) a  $\varphi$  of the form  $\varphi(z) = B \left( 1 + K^{\frac{1}{\alpha}} z^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}}$ ,  $B > 0$ . In particular, taking any pair  $(\rho, \gamma) \in (0, 1) \times \mathbb{R}_{++}$  such that  $((1 - \rho) / \rho) \gamma^{\frac{\alpha-1}{\alpha}} = K^{\frac{1}{\alpha}}$  (there is a continuum of possibilities) and putting  $B = \rho^{\frac{\alpha}{\alpha-1}}$  conveniently yields  $\varphi(z) = \left( \rho + (1 - \rho) (\gamma z)^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}}$  and

$$u(c, m) = g_\sigma \left( \left( \rho c^{\frac{\alpha-1}{\alpha}} + (1 - \rho) (\gamma m)^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}} \right). \quad (15)$$

Note that (15) is only defined for  $\alpha \in \mathbb{R}_+ \setminus \{0, 1\}$ . This domain restriction must be kept in mind regarding the results to be proved shortly. Because Fischer's CES utility (11) and Asako's

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<sup>7</sup>As shown in Appendix B, this is indeed enough to assure us of the validity of the assumptions made on  $u$ .

CES utility (12) are limiting cases of the class defined by (15) (when the elasticity of substitution  $\alpha$  approaches 1 and 0 respectively, as shown in Arrow et al. 1961, or Appendix C), but do not properly belong to it, we shall refer to this class as the CES\* class.

A function belonging to the CES\* class differs from Fischer's and Asako's also in another respect: it may not be CRRA. We actually have the following

**Lemma 1** *Given the CES\* utility function  $u$  in (15), the following are equivalent: (i)  $\sigma = 1/\alpha$ ; (ii)  $\sigma_R = 1/\alpha$ ; (iii)  $u$  is CRRA; and (iv)  $u$  is (strongly) separable.*

**Proof.** As shown in Appendix D,  $u$ 's coefficient of relative risk aversion is

$$\sigma_R(c, m) := -c \frac{u_{11}(c, m)}{u_1(c, m)} = \frac{c^{\frac{\alpha-1}{\alpha}} \sigma + K^{\frac{1}{\alpha}} m^{\frac{\alpha-1}{\alpha}} \frac{1}{\alpha}}{c^{\frac{\alpha-1}{\alpha}} + K^{\frac{1}{\alpha}} m^{\frac{\alpha-1}{\alpha}}}, \quad (16)$$

that is, a weighted average of  $\sigma$  and  $1/\alpha$ . So (i)  $\implies$  (ii)  $\implies$  (iii) is clear. Also, since the weights in the average  $\sigma_R$  are nonconstant with respect to  $(c, m)$  (because  $\alpha \neq 1$ ), (iii)  $\implies$  (i). Finally, it is clear from (15) that a CES\*-class  $u$  can only be separable in the  $\sigma \neq 1$  case, more specifically when the relevant exponent,  $(\alpha/(\alpha-1))(1-\sigma)$ , simplifies to 1. This is equivalent to (i). ■

Before presenting our results, it should be noted that although  $r$  was defined only in the steady state ( $r = \theta + \delta$ ), we could also have chosen to take it endogenously, in accordance with the equilibrium equation (7), as  $u_2/u_1$ . The same math leading to (14), and which can be checked in Appendix D, would determine the path for  $r$  as a function of those for  $c$  and  $m$ :

$$r = \left( K \frac{c}{m} \right)^{\frac{1}{\alpha}} = \left( K \frac{c^* + e^{\lambda(\theta)t} (c_0 - c^*)}{m^* + e^{\lambda(\theta)t} (m_0 - m^*)} \right)^{\frac{1}{\alpha}}, \quad (17)$$

already considering the convergent solution to the linear system (8).<sup>8</sup>

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<sup>8</sup> Alternatively, (1) would give  $r = f'(k) + \pi$ , so that the approximate path for  $r$  near the  $(c^*, m^*, k^*)$  steady state could be written as  $r = f'(k^* + e^{\lambda(\theta)t} (k_0 - k^*)) + \pi$ .

Equation (17) gives the mirror-image of the monetary experience described by Fischer. It regards the path followed by the nominal interest rate. To a certain extent it provides a link between the experiences on which one used to focus in the 80's (and which we analyze here), based on an exogenous control of the money supply; and those most common in recent years, which concentrate on the indirect control of interest rates by the monetary authorities.

### 3 Results

Our goal here is to find  $\lambda'(\theta)$ , under the utility specification (15). The characteristic polynomial (9) leads to a characteristic equation  $\Psi(\lambda, \theta) = 0$ , where (as shown in Appendix D)

$$\Psi(\lambda, \theta) := \left( Kr + r^\alpha - \frac{K + \alpha \sigma r^{\alpha-1}}{\sigma} \lambda \right) (\sigma (\lambda^2 - \delta \lambda) + f''(k^*) c^*) + f''(k^*) c^* \frac{K(1-\sigma)(1-\alpha\sigma)}{\sigma} \lambda \quad (18)$$

$$= ((Kr + r^\alpha - \alpha r^{\alpha-1} \lambda) (\lambda^2 - \delta \lambda) + f''(k^*) c^* K \alpha \lambda) \sigma - K \lambda (\lambda^2 - \delta \lambda) + f''(k^*) c^* (Kr + r^\alpha - \alpha r^{\alpha-1} \lambda - K(1+\alpha) \lambda), \quad (19)$$

and  $r := \theta + \delta$ .

Unless the contrary is stated, we shall be looking at the first form above, (18). Since  $\Psi(0, \theta) = f''(k^*) c^* (Kr + r^\alpha) < 0$  and  $\Psi(\cdot, \theta) = 0$  has  $\lambda(\theta)$  as its only negative root, we cannot have  $\Psi_1(\lambda(\theta), \theta) > 0$ . Moreover, from  $\lambda(\theta)$ 's multiplicity being 1, we are assured it isn't an inflection point of the function  $\Psi(\cdot, \theta)$ , so that  $\Psi_1(\lambda(\theta), \theta) < 0$  indeed. Since  $\lambda'(\theta) = -(\Psi_2/\Psi_1)(\lambda(\theta), \theta)$  (Implicit Function Theorem), we then have  $\text{sgn}(\lambda'(\theta)) = \text{sgn}(\Psi_2(\lambda(\theta), \theta))$ .

Now,  $\Psi_2(\lambda, \theta) = (K + \alpha r^{\alpha-1} + \alpha(1-\alpha)r^{\alpha-2}\lambda) (\sigma (\lambda^2 - \delta \lambda) + f''(k^*) c^*)$ . The sign of the first factor is, in principle, undetermined (unless  $\alpha > 1$ ). As for  $\sigma (\lambda(\theta)^2 - \delta \lambda(\theta)) + f''(k^*) c^*$ , (18) implies (from  $\Psi(\lambda(\theta), \theta) = 0$  and  $\lambda(\theta) < 0$ ) that it shares its sign with  $-(1-\sigma)(1-\alpha\sigma)$ .

Up to this point, our main conclusion is that

$$\operatorname{sgn}(\lambda'(\theta)) = -\operatorname{sgn}(\sigma - 1) \operatorname{sgn}\left(\sigma - \frac{1}{\alpha}\right) \operatorname{sgn}(K + \alpha r^{\alpha-1} + \alpha(1 - \alpha)r^{\alpha-2}\lambda(\theta)), \quad (20)$$

which immediately yields

**Proposition 1** *Suppose the elasticity of money demand is greater than one. Then*

$$\operatorname{sgn}(\lambda'(\theta)) = -\operatorname{sgn}(\sigma - 1) \operatorname{sgn}\left(\sigma - \frac{1}{\alpha}\right). \quad (21)$$

This proposition illustrates how easy it is, in contrast with Fischer (1979), to generate a negative correlation between the growth rate of money supply and the rate of capital accumulation ( $\lambda'(\theta) > 0$ ), within the CES\* class. According to it, if  $\alpha > 1$ , this will happen whenever  $\sigma \in (1/\alpha, 1)$ . This case, however, should find a more limited use in real-world applications.

We now focus on the more realistic  $\alpha < 1$  case. The following developments shall make it possible to replace the complicated condition in the last term in (20), depending on the unknown value  $\lambda(\theta)$ , with a very determinate threshold for  $\sigma$ , depending only on the parameters of the problem. Let  $\bar{\lambda}$  be the value  $\lambda(\theta)$  would have to take on so that the final term in (20),  $K + \alpha r^{\alpha-1} + \alpha(1 - \alpha)r^{\alpha-2}\lambda(\theta)$ , equaled 0 – that is,

$$\bar{\lambda} := -\frac{K + \alpha r^{\alpha-1}}{\alpha(1 - \alpha)r^{\alpha-2}}. \quad (22)$$

Therefore  $K + \alpha r^{\alpha-1} + \alpha(1 - \alpha)r^{\alpha-2}\lambda(\theta) \lesseqgtr 0 \iff \lambda(\theta) \lesseqgtr \bar{\lambda} \iff \Psi(\bar{\lambda}, \theta) \lesseqgtr 0$ , since  $\Psi_1(\lambda(\theta), \theta) < 0$  and  $\lambda(\theta)$  is the only negative root of  $\Psi(\cdot, \theta)$ .

If all parameters besides  $\sigma$  are fixed, and so is  $\lambda$  (at some negative value), (19) presents  $\Psi$  as a

simple increasing affine function of  $\sigma$ .<sup>9</sup> Let  $s(\lambda)$  represent the (possibly nonpositive) value  $\sigma$  would have to take on in order to make  $\Psi$  vanish:

$$s(\lambda) = \frac{K\lambda(\lambda^2 - \delta\lambda) - f''(k^*)c^*(Kr + r^\alpha - \alpha r^{\alpha-1}\lambda - K(1 + \alpha)\lambda)}{(Kr + r^\alpha - \alpha r^{\alpha-1}\lambda)(\lambda^2 - \delta\lambda) + f''(k^*)c^*K\alpha\lambda}. \quad (23)$$

Now  $\Psi(\bar{\lambda}, \theta) \leq 0 \iff \sigma \leq s(\bar{\lambda})$ , and putting the pieces together gives

**Proposition 2** *Suppose the elasticity of real balances with respect to the nominal interest rate is lower than one. Then*

$$\text{sgn}(\lambda'(\theta)) = -\text{sgn}(\sigma - 1)\text{sgn}\left(\sigma - \frac{1}{\alpha}\right)\text{sgn}(\sigma - s(\bar{\lambda})), \quad (24)$$

where the function  $s : \mathbb{R}_{--} \rightarrow \mathbb{R}$  is defined in (23), and  $\bar{\lambda}$  in (22).

In the following, we denote the possibility that  $\lambda(\theta) = \bar{\lambda}$  (particular of the  $\alpha < 1$  case) or, equivalently,  $\sigma = s(\bar{\lambda})$ , simply by  $P$ . The next proposition, our general transition-path superneutrality result, is valid for any  $\alpha \in \mathbb{R}_+ \setminus \{0, 1\}$ :

**Proposition 3** *Within the CES\* class, transition-path superneutrality occurs if, and only if, either (i)  $\sigma = 1$  (logarithmic utility); or (ii)  $\sigma = 1/\alpha$  (or either one of the other three equivalent conditions stated in Lemma 1); or (iii)  $P$ .*

Among these three cases, (i) is the only one mentioned in Fischer (1979). Cases (ii) and (iii) show that under more general utilities there are other possibilities under which one may obtain  $\lambda'(\theta) = 0$  as well.

If one wishes to compare our results with those of Asako and Fischer, the behavior of  $s(\bar{\lambda})$  when  $\alpha \downarrow 0$  and when  $\alpha \uparrow 1$  must be studied. As shown in Appendix E, both limits are negative,

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<sup>9</sup>Equations (5) and (6) show that the steady-state values  $k^*$  and  $c^*$  have no dependence on  $\sigma$ .

so  $\sigma - s(\bar{\lambda}) > 0$ , and (24) becomes merely (21). We also have (21) when  $\alpha \downarrow 1$ , from Proposition 1. If  $\sigma = 1$ , our superneutrality result evidently concurs with theirs, so we proceed with the  $\sigma \neq 1$  case. Since we cannot have  $\sigma = 1/\alpha$  ( $\alpha$  is taking part in a strictly monotonic convergent process),  $\lambda'(\theta)$  cannot vanish, so that  $\text{sgn}$  is continuous at  $\lambda'(\theta)$ . Therefore  $\text{sgn}(\lambda'(\theta)) = \lim \text{sgn}(\lambda'(\theta)) = -\text{sgn}(\sigma - 1) \lim \text{sgn}(\sigma - 1/\alpha)$ , where the limit can be for either  $\alpha \rightarrow 0_+$  or  $\alpha \rightarrow 1$ . In the first case,  $\text{sgn}(\lambda'(\theta)) = \text{sgn}(\sigma - 1)$ , and in the second,  $\text{sgn}(\lambda'(\theta)) = -(\text{sgn}(\sigma - 1))^2 = -1$ . Thus, although our math was done for  $\alpha \notin \{0, 1\}$ , our conclusions, if taken to the appropriate limits, generalize the results obtained by Asako and Fischer for  $\alpha = 0$  and  $\alpha = 1$ .

In order to visualize these results, we take the parameter values:  $\alpha = 0.5$ ,  $K = 0.05$ ,  $\delta = 0.09$ ,  $n = 0.01$ ,  $\theta = 0.03$  and  $f(k) = 0.56k^{0.38}$ .<sup>10</sup> Using (5), (6) and (14), we get  $k^* \approx 3.38$ ,  $c^* \approx 0.86$  and  $m^* \approx 0.12$ , steady-state levels in conformity with those presented by developed economies. We now let  $\alpha$  free, and concentrate our analysis on the  $\alpha \times \sigma$  plane.<sup>11</sup> Figure 1 summarizes our conclusions.

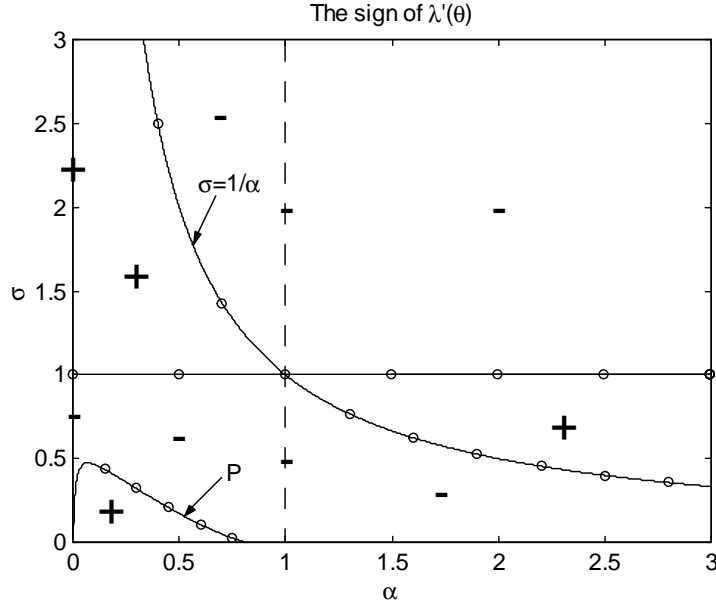


Figure 1

<sup>10</sup>The first two values were calibrated for the U.S. by Lucas (2000, pp. 250 and 258). The elasticity of the production function is consistent with estimates for the OECD of capital's share of income in the traditional Solow model (Mankiw et al. 1992, p. 414).

<sup>11</sup>From equations (5), (6) and (14), this approach leaves  $k^*$  and  $c^*$  fixed, but not  $m^*$ .

The signs on the  $\alpha = 0$  and the  $\alpha = 1$  vertical lines allude to Asako's and Fischer's results. The little circles on the  $\sigma = 1$ , the  $\sigma = 1/\alpha$  and the  $P$  curves remind us that over these sets of points  $\lambda'(\theta) = 0$ , the transition-path superneutrality result. Fischer's superneutrality result identifies the single point  $(1, 1)$ , for which utility is logarithmic, separable, and the interest-rate elasticity of money demand is equal to one. Concerning the relationship between capital accumulation and money supply, regions and borders marked with a plus sign are all contrasting with Fischer's results. The ray  $\{0\} \times (1, +\infty)$  corresponds to the counterexample provided by Asako to Fischer's results.

Some general remarks may be useful in the understanding of the results displayed in Figure 1. The  $\sigma = 1/\alpha$  curve is really the  $u_{12} = 0$  curve (see Appendix D), with the region above it corresponding to  $u_{12} < 0$  and the region below it to  $u_{12} > 0$ . The importance of  $u_{12}$  is captured in Fischer's (1979, p. 1439) conjecture that "the effect on capital accumulation results from the influence of holdings of real balances on the marginal utility of consumption", although in that work (in which  $\alpha = 1$ ), as acknowledged by Fischer, it is not clear how this would be consistent with his finding that  $\lambda'(\theta) < 0$  for both  $\sigma < 1$  and  $\sigma > 1$ . Figure 1 gives the answer: following along any vertical line  $\alpha = \bar{\alpha}$ , both curves  $\sigma = 1$  and  $\sigma = 1/\alpha$  do in fact correspond to a change in sign for  $\lambda'(\theta)$ . But if these curves happen to cut the vertical line in the exact same place (which is what happens in the  $\bar{\alpha} = 1$  case analyzed by Fischer), then the changes in sign will end up neutralizing each another, looking like if none ever took place. Through the dependence of capital accumulation on the influence of money holdings on the marginal utility of consumption one also understands why, in particular,  $\lambda'(\theta) = 0$  along the  $\sigma = 1/\alpha$  curve or when  $\sigma = 1$ . Indeed, in both of these cases  $u_{12} = 0$ .

Tempting explanations for the signs which emerge in Figure 1 may be misleading. Take for instance the region  $\alpha < 1$  and below the  $\sigma = 1/\alpha$  curve. Suppose one wishes to explain the positive correlation between the rate of capital accumulation and money growth in the area above curve



$P$  by arguing that increasing  $\theta$  would lead to higher nominal interest rates, lower real balances and (because  $u_{12} > 0$  in this region), a lower marginal utility of consumption. That this argument cannot be correct is shown by the fact that it should also hold below curve  $P$ , which it does not.

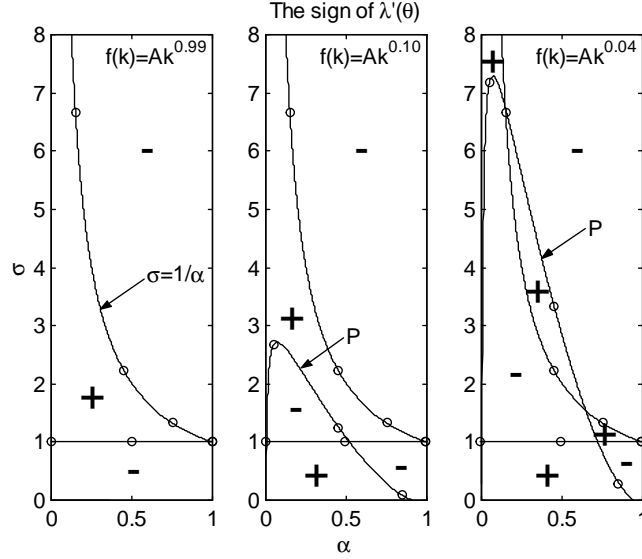


Figure 2

Figure 2 shows, through (possibly unrealistic) changes exclusively in the elasticity of the production function (other parameter values kept as before), the other possible qualitative behaviors of the sign of  $\lambda'(\theta)$ . It focuses on  $\alpha \in (0, 1)$ , since the  $\alpha > 1$  case as depicted in Figure 1 remains unchanged (Proposition 1), the same happening with the  $\alpha = 0$  and the  $\alpha = 1$  cases. The graph at the left illustrates that, for very large elasticities (0.98 and higher), possibility  $P$  may not arise at all, since  $s(\bar{\lambda}) < 0, \forall \alpha \in (0, 1)$ . Concerning the other two graphs, it should be noted that there is a region (to which a minus sign should be assigned) that cannot be depicted: a very thin strip for  $\alpha$  very close to 0, and  $\sigma$  lower than 1 but greater than  $s(\bar{\lambda})$ .<sup>12</sup>

<sup>12</sup>The region exists because  $\lim_{\alpha \rightarrow 0+} s(\bar{\lambda})$  is negative (see Appendix E).

## 4 Conclusion

This paper has extended Fischer's (1979) and Asako's (1983) results on the sign of the correlation between the growth rate of money and the rate of capital accumulation on the transition path by dealing with a more general class of consumer preferences under which the elasticity of real balances with respect to the nominal interest rate, rather than necessarily being equal to one or zero, can assume any other positive real value. We have characterized the sign of that correlation as a function of the parameters of the problem, showing that it can easily be negative (the plus signs in Figure 1), contrary to Fischer's prediction. We have also shown how Fischer's and Asako's contrasting results can both be understood as particular cases of this more encompassing analysis.

## References

- [1] Arrow, Kenneth J., Hollis B. Chenery, Bagicha S. Minhas, and Robert M. Solow. (1961). "Capital-Labor Substitution and Economic Efficiency." *The Review of Economics and Statistics* 43, 225-250.
- [2] Asako, Kazumi. (1983). "The Utility Function and the Superneutrality of Money on the Transition Path." *Econometrica* 51, 1593-1596.
- [3] Cysne, Rubens P., and David Turchick. (2009). "On the Integrability of Money-Demand Functions by the Sidrauski and the Shopping-Time Models." *Journal of Banking & Finance* 33, 1555-1562.
- [4] Fischer, Stanley. (1979). "Capital Accumulation on the Transition Path in a Monetary Optimizing Model." *Econometrica* 47, 1433-1439.

- [5] Jones, Barry, Gabriel Asaftei, and Lian Wang. (2004). "Welfare Cost of Inflation in a General Equilibrium Model with Currency and Interest-Bearing Deposits." *Macroeconomic Dynamics* 8, 493-517.
- [6] Lucas, Robert E., Jr. (2000). "Inflation and Welfare." *Econometrica* 68, 247-274.
- [7] Mankiw, N. Gregory, David Romer, and David N. Weil. (1992). "A Contribution to the Empirics of Economic Growth." *The Quarterly Journal of Economics* 107, 407-437.
- [8] McCallum, Bennett T. (2008). "How Important is Money in the Conduct of Monetary Policy? A Comment." *Journal of Money, Credit and Banking* 40, 1783-1790.
- [9] Nelson, Edward (2008). "Why Money Growth Determines Inflation in the Long Run: Answering the Woodford Critique." *Journal of Money, Credit and Banking* 40, 1791-1814.
- [10] Sidrauski, Miguel. (1967). "Rational Choice and Patterns of Growth in a Monetary Economy." *American Economic Review* 57, 534-544.
- [11] Woodford, Michael. (2008). "How Important is Money in the Conduct of Monetary Policy?" *Journal of Money, Credit and Banking* 40, 1561-1598.

## Appendix A

We wish to show that  $D$ 's characteristic polynomial is (9) indeed. One has

$$\begin{aligned} & -\lambda^3 + (\operatorname{tr} D) \lambda^2 - (D_{11}D_{22} - D_{12}D_{21} + D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33} - D_{23}D_{32}) \lambda + \det D \\ = & -\lambda^3 + (\operatorname{tr} D) \lambda^2 - (D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33}) \lambda + \det D, \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr} D &= \delta + \eta, \\ D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33} &= D_{33}(D_{11} + D_{22}) - D_{13}D_{31} = \delta\eta - f'' \frac{u_1 + mu_{12}}{u_{11}}, \\ \det D &= -mf''J_2 \left( m \frac{u_{12}}{u_{11}} - \frac{u_1 + mu_{12}}{u_{11}} \right) = mf''J_2 \frac{u_1}{u_{11}}, \end{aligned}$$

where  $\eta := m(J_1u_{12}/u_{11} - J_2)$ , and the determinant was calculated by application of Laplace's rule to  $D$ 's third row. So the characteristic polynomial is

$$-\lambda^3 + (\delta + \eta) \lambda^2 - \left( \delta\eta - f'' \frac{u_1 + mu_{12}}{u_{11}} \right) \lambda + mf''J_2 \frac{u_1}{u_{11}},$$

and it is easy to compare coefficient by coefficient to see that this expression coincides with (9).

## Appendix B

Here we check that the assumptions made on  $u$  at the beginning of Section 2 are satisfied by the functional form (13). Let  $v : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_{++}$  be given by  $v(c, m) = c\varphi(m/c)$ , so that  $u = g_\sigma \circ v$ .

Function  $g_\sigma$  is obviously increasing and concave: for any  $x \in \mathbb{R}_{++}$ ,  $g'_\sigma(x) = x^{-\sigma} > 0$  and  $g''_\sigma(x) = -\sigma x^{-\sigma-1} < 0$ . As for  $v$ , let  $(c, m) \in \mathbb{R}_{++}^2$  and  $z := m/c$ . Then  $v_1(c, m) = \varphi(z) - z\varphi'(z)$ , which is positive (the strict concavity and nonnegativity of  $\varphi$  give  $\varphi'(z)(0 - z) > \varphi(0) -$

$\varphi(z) \geq -\varphi(z)$  and  $v_2(c, m) = \varphi'(z) > 0$ . Using that  $z_1 = -z/c$ , we also have  $v_{11}(c, m) = (\varphi'(z) - \varphi'(z) - z\varphi''(z))z_1 = (z^2/c)\varphi''(z) < 0$ ,  $v_{22}(c, m) = (1/c)\varphi''(z) < 0$ ,  $v_{12}(c, m) = v_{21}(c, m) = (-z/c)\varphi''(z)$  and  $(v_{11}v_{22} - v_{12}^2)(c, m) = 0$ , so that  $v$  is also concave. Therefore,  $u$  is concave.

Although many of the following calculations wouldn't have to be carried through since it is only the sign of these expressions that matters for now, we do it since these expressions shall be needed in Appendix D. All the derivatives of  $v$  below are evaluated at  $(c, m)$ , and all the derivatives of  $g_\sigma$  at  $v(c, m)$ :

$$\begin{aligned}
u_1 &= g'_\sigma v_1 > 0 \\
&= (c\varphi(z))^{-\sigma} (\varphi(z) - z\varphi'(z)), \\
u_2 &= g'_\sigma v_2 > 0 \\
&= (c\varphi(z))^{-\sigma} \varphi'(z), \\
u_{11} &= g''_\sigma v_1^2 + g'_\sigma v_{11} < 0 \\
&= (c\varphi(z))^{-\sigma-1} \left( -\sigma (\varphi(z) - z\varphi'(z))^2 + z^2 \varphi(z) \varphi''(z) \right), \\
u_{22} &= g''_\sigma v_2^2 + g'_\sigma v_{22} < 0 \\
&= (c\varphi(z))^{-\sigma-1} \left( -\sigma \varphi'(z)^2 + \varphi(z) \varphi''(z) \right), \\
u_{12} &= g''_\sigma v_1 v_2 + g'_\sigma v_{12} = (c\varphi(z))^{-\sigma-1} \left( -\sigma \varphi'(z) (\varphi(z) - z\varphi'(z)) - z\varphi(z) \varphi''(z) \right).
\end{aligned}$$

Also, for  $i \in \{1, 2\}$ ,

$$J_i = \left( \frac{u_2}{u_1} \right)_i = \frac{d}{dz} \left( \frac{\varphi'(z)}{\varphi(z) - z\varphi'(z)} \right) z_i,$$

so that

$$\begin{aligned}
J_1 &= \frac{\varphi''(z)(\varphi(z) - z\varphi'(z)) - \varphi'(z)(-z\varphi''(z))}{(\varphi(z) - z\varphi'(z))^2} \left(-\frac{z}{c}\right) = -\frac{z}{c} \frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} > 0, \\
J_2 &= \frac{1}{c} \frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} = -\frac{J_1}{z} < 0.
\end{aligned}$$

## Appendix C

Here we confirm that the instantaneous utility functions taken in Fischer (1979) and Asako (1983) are limiting cases of ours. More specifically, we wish to check that the function  $v : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_{++}$  given by  $v(c, m) = \left(\rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}$  is such that  $\lim_{\alpha \rightarrow 1} v(c, m) = c^\rho (\gamma m)^{1-\rho}$  and  $\lim_{\alpha \rightarrow 0+} v(c, m) = \min(c, \gamma m)$ . In fact,

$$\begin{aligned}
\lim_{\alpha \rightarrow 1} v(c, m) &= \exp \lim_{\alpha \rightarrow 1} \frac{\log \left( \rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}} \right)}{\frac{\alpha-1}{\alpha}} \\
&= \exp \lim_{\alpha \rightarrow 1} \frac{\left( \rho c^{\frac{\alpha-1}{\alpha}} \log c + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}} \log(\gamma m) \right) \frac{d}{d\alpha} \left( \frac{\alpha-1}{\alpha} \right)}{\left( \rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}} \right) \frac{d}{d\alpha} \left( \frac{\alpha-1}{\alpha} \right)} \\
&= \exp(\rho \log c + (1-\rho) \log(\gamma m)) = c^\rho (\gamma m)^{1-\rho},
\end{aligned}$$

where the second equality used l'Hôpital's rule, and

$$\begin{aligned}
\lim_{\alpha \rightarrow 0+} v(c, m) &= \lim_{n \rightarrow +\infty} \left( \rho c^{-n} + (1-\rho)(\gamma m)^{-n} \right)^{-\frac{1}{n}} \\
&= \lim_{n \rightarrow +\infty} \frac{1}{\left( \rho \left( \frac{1}{c} \right)^n + (1-\rho) \left( \frac{1}{\gamma m} \right)^n \right)^{\frac{1}{n}}} \\
&= \frac{1}{\max \left( \frac{1}{c}, \frac{1}{\gamma m} \right)} = \min(c, \gamma m),
\end{aligned}$$

where the third equality used the fact that, if  $0 \leq a \leq b$  and  $x_n := (\rho a^n + (1-\rho)b^n)^{\frac{1}{n}}$ , then  $\lim_{n \rightarrow +\infty} x_n = b$ , since  $x_n \leq (\rho b^n + (1-\rho)b^n)^{\frac{1}{n}} = b$  and  $\lim_{n \rightarrow +\infty} x_n \geq \lim_{n \rightarrow +\infty} ((1-\rho)b^n)^{\frac{1}{n}} =$

b.

## Appendix D

Here we work out the derivatives of  $u$  given by (15) that are necessary for the derivation of  $r$  in (17), the coefficient of relative risk aversion  $\sigma_R$  in (16), and  $\Psi$  in (18) and (19). Given the developments

made in Appendix B, we can forget about  $u$  and focus on  $\varphi$  only. As mentioned in Section 2, we

should use  $\varphi(z) = \left(\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{1}{\alpha-1}}$ . Therefore  $\varphi'(z) = \left(\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{1}{\alpha-1}} (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} = (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (\varphi(z)/z)^{\frac{1}{\alpha}}$ ,  $\varphi(z) - z\varphi'(z) = \varphi(z)^{\frac{1}{\alpha}} \left(\varphi(z)^{\frac{\alpha-1}{\alpha}} - (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{1-\frac{1}{\alpha}}\right) = \rho\varphi(z)^{\frac{1}{\alpha}}$ , so that  $\varphi''(z) = (1/\alpha)(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (\varphi(z)/z)^{\frac{1}{\alpha}-1} (- (\varphi(z) - z\varphi'(z))/z^2) = -(1/\alpha)\rho(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-1-\frac{1}{\alpha}} \varphi(z)^{\frac{2-\alpha}{\alpha}}$ .

Using the expressions found in the previous appendix,

$$\begin{aligned}
u_1 &= \rho(c\varphi(z))^{-\sigma} \varphi(z)^{\frac{1}{\alpha}}, \\
u_2 &= (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (c\varphi(z))^{-\sigma} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{1}{\alpha}} \\
u_{11} &= (c\varphi(z))^{-\sigma-1} \left( -\sigma\rho^2\varphi(z)^{\frac{2}{\alpha}} - \frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{1-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}} \right) \\
&= -\frac{\rho}{\alpha} (c\varphi(z))^{-\sigma-1} \varphi(z)^{\frac{2}{\alpha}} \left( \alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}} \right), \\
u_{12} &= (c\varphi(z))^{-\sigma-1} \left( -\sigma\rho(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}} + \frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}} \right) \\
&= \frac{\rho(1-\rho)(1-\alpha\sigma)\gamma^{\frac{\alpha-1}{\alpha}}}{\alpha} (c\varphi(z))^{-\sigma-1} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}, \\
J_1 &= -\frac{z}{c} \frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} = \frac{1}{c} \frac{\frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}}{\rho^2\varphi(z)^{\frac{2}{\alpha}}} = \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}}}{\rho\alpha} \frac{1}{c}, \\
J_2 &= -\frac{J_1}{z} = -\frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-1-\frac{1}{\alpha}}}{\rho\alpha} \frac{1}{c}.
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{u_2}{u_1} &= \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}z^{-\frac{1}{\alpha}}}{\rho}, \\ \frac{u_1}{u_{11}} &= -\alpha \frac{c\varphi(z)}{\varphi(z)^{\frac{1}{\alpha}}\left(\alpha\sigma\rho + (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}\right)} = -\alpha c \frac{\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}}{\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}},\end{aligned}$$

so that (and now using  $K^{\frac{1}{\alpha}} = ((1-\rho)/\rho)\gamma^{\frac{\alpha-1}{\alpha}}$ )

$$\begin{aligned}r &: = \frac{u_2}{u_1} = K^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}} = \left(K\frac{c}{m}\right)^{\frac{1}{\alpha}}, \\ \sigma_R &: = -c\frac{u_{11}}{u_1} = \frac{1}{\alpha} \frac{\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}}{\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}} = \frac{c^{\frac{\alpha-1}{\alpha}}\sigma + K^{\frac{1}{\alpha}}m^{\frac{\alpha-1}{\alpha}}\frac{1}{\alpha}}{c^{\frac{\alpha-1}{\alpha}} + K^{\frac{1}{\alpha}}m^{\frac{\alpha-1}{\alpha}}}.\end{aligned}$$

Also, at the steady state (in order to find the characteristic polynomial), we have (where  $c$  and  $m$  are short for  $c^*$  and  $m^*$ )

$$\begin{aligned}\frac{u_1}{u_{11}} &= -\alpha c \frac{1 + K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}}{\alpha\sigma + K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}} = -\alpha c \frac{1 + rz}{\alpha\sigma + rz} = -\alpha c \frac{K + r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}}, \\ \frac{u_{12}}{u_{11}} &= -\frac{(1-\rho)(1-\alpha\sigma)\gamma^{\frac{\alpha-1}{\alpha}}z^{-\frac{1}{\alpha}}}{\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}} = -\frac{K^{\frac{1}{\alpha}}(1-\alpha\sigma)z^{-\frac{1}{\alpha}}}{\alpha\sigma + K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}} = -\frac{(1-\alpha\sigma)r}{\alpha\sigma + rz} = -\frac{(1-\alpha\sigma)r^{\alpha}}{K + \alpha\sigma r^{\alpha-1}}, \\ \frac{u_1 + mu_{12}}{u_{11}} &= -\frac{\alpha c(K + r^{\alpha-1}) + (1-\alpha\sigma)mr^{\alpha}}{K + \alpha\sigma r^{\alpha-1}} = -c \frac{\alpha(K + r^{\alpha-1}) + K(1-\alpha\sigma)}{K + \alpha\sigma r^{\alpha-1}} \\ &= -c \frac{(1+\alpha-\alpha\sigma)K + \alpha r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}}, \\ m\left(J_1\frac{u_{12}}{u_{11}} - J_2\right) &= mJ_1\left(\frac{u_{12}}{u_{11}} + \frac{1}{z}\right) = \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}z^{1-\frac{1}{\alpha}}\left(-\frac{(1-\alpha\sigma)r^{\alpha}}{K + \alpha\sigma r^{\alpha-1}} + \frac{r^{\alpha}}{K}\right) \\ &= \frac{K^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}}}{\alpha}z\frac{r^{\alpha}}{K}\left(1 - \frac{K(1-\alpha\sigma)}{K + \alpha\sigma r^{\alpha-1}}\right) = \frac{r}{\alpha}\frac{K + \alpha\sigma r^{\alpha-1} - K + K\alpha\sigma}{K + \alpha\sigma r^{\alpha-1}} = \frac{\sigma(Kr + r^{\alpha})}{K + \alpha\sigma r^{\alpha-1}}, \\ mJ_2\frac{u_1}{u_{11}} &= -m\frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}z^{-1-\frac{1}{\alpha}}\left(-\alpha c\frac{K + r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}}\right) = K^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}}c\frac{K + r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}} = c\frac{Kr + r^{\alpha}}{K + \alpha\sigma r^{\alpha-1}}.\end{aligned}$$



Plugging these expressions into (9) yields

$$\begin{aligned}
& (\lambda^2 - \delta\lambda) \left( -\lambda + \frac{\sigma(Kr + r^\alpha)}{K + \alpha\sigma r^{\alpha-1}} \right) + f'' \left( -c \frac{(1 + \alpha - \alpha\sigma)K + \alpha r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}} \lambda + c \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} \right) \\
= & \sigma(\lambda^2 - \delta\lambda) \left( -\frac{\lambda}{\sigma} + \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} \right) + f'' c \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} - f'' c \frac{(1 + \alpha - \alpha\sigma)K + \alpha r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}} \lambda \\
= & (\sigma(\lambda^2 - \delta\lambda) + f'' c) \left( -\frac{\lambda}{\sigma} + \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} \right) + f'' c \frac{\lambda}{\sigma} - f'' c \frac{(1 + \alpha - \alpha\sigma)K + \alpha r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}} \lambda \\
= & \left( \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} - \frac{\lambda}{\sigma} \right) (\sigma(\lambda^2 - \delta\lambda) + f'' c) + f'' c \frac{\frac{1}{\sigma}(K + \alpha\sigma r^{\alpha-1}) - (1 + \alpha - \alpha\sigma)K - \alpha r^{\alpha-1}}{K + \alpha\sigma r^{\alpha-1}} \lambda \\
= & \left( \frac{Kr + r^\alpha}{K + \alpha\sigma r^{\alpha-1}} - \frac{\lambda}{\sigma} \right) (\sigma(\lambda^2 - \delta\lambda) + f'' c) + f'' c \frac{K(1 - \sigma)(1 - \alpha\sigma)}{\sigma} \lambda = \frac{\Psi(\lambda, \theta)}{K + \alpha\sigma r^{\alpha-1}},
\end{aligned}$$

if  $\Psi$  is defined as in (18). From this first form of  $\Psi$ , it is a trivial task to obtain the second, (19):

$$\begin{aligned}
& \left( Kr + r^\alpha - \frac{K + \alpha\sigma r^{\alpha-1}}{\sigma} \lambda \right) (\sigma(\lambda^2 - \delta\lambda) + f'' c) + f'' c \frac{K(1 - \sigma)(1 - \alpha\sigma)}{\sigma} \lambda \\
= & \left( Kr + r^\alpha - \alpha r^{\alpha-1} \lambda - \frac{K}{\sigma} \lambda \right) \sigma(\lambda^2 - \delta\lambda) + f'' c \left( Kr + r^\alpha - \alpha r^{\alpha-1} \lambda - \frac{K}{\sigma} \lambda \right) + f'' c \frac{K(1 - \sigma - \alpha\sigma + \alpha\sigma^2)}{\sigma} \lambda \\
= & (Kr + r^\alpha - \alpha r^{\alpha-1} \lambda) (\lambda^2 - \delta\lambda) \sigma - K\lambda (\lambda^2 - \delta\lambda) + f'' c (Kr + r^\alpha - \alpha r^{\alpha-1} \lambda) + f'' c K(-1 - \alpha + \alpha\sigma) \lambda \\
= & ((Kr + r^\alpha - \alpha r^{\alpha-1} \lambda) (\lambda^2 - \delta\lambda) + f'' c K \alpha \lambda) \sigma - K\lambda (\lambda^2 - \delta\lambda) + f'' c (Kr + r^\alpha - \alpha r^{\alpha-1} \lambda - K(1 + \alpha) \lambda).
\end{aligned}$$

## Appendix E

Here we are concerned with the side limits of  $s(\bar{\lambda})$  when  $\alpha \downarrow 0$  and when  $\alpha \uparrow 1$ . From (22), we

immediately get  $\lim_{\alpha \rightarrow 0+} \bar{\lambda} = -\infty$  and  $\lim_{\alpha \rightarrow 1-} \bar{\lambda} = -\infty$ . It follows that

$$\begin{aligned}
\lim & \frac{Kr + r^\alpha - \alpha r^{\alpha-1} \bar{\lambda} - K(1 + \alpha) \bar{\lambda}}{(Kr + r^\alpha - \alpha r^{\alpha-1} \bar{\lambda}) (\bar{\lambda}^2 - \delta \bar{\lambda}) + f''(k^*) c^* K \alpha \bar{\lambda}} = \lim \frac{Kr + r^\alpha - \alpha r^{\alpha-1} \bar{\lambda}}{(Kr + r^\alpha - \alpha r^{\alpha-1} \bar{\lambda}) (\bar{\lambda}^2 - \delta \bar{\lambda}) + f''(k^*) c^* K \alpha \bar{\lambda}} \\
& \lim \frac{K(1 + \alpha)}{(Kr + r^\alpha - \alpha r^{\alpha-1} \bar{\lambda}) (\bar{\lambda} - \delta) + f''(k^*) c^* K \alpha} = 0 - 0 = 0,
\end{aligned}$$

no matter which side limit is being considered. Additionally, from (5) and (6) we know that neither  $k^*$  nor  $c^*$  depend on  $\alpha$ . We thus have, from (23) and (22),

$$\begin{aligned}
\lim s(\bar{\lambda}) &= \lim \frac{K\bar{\lambda}(\bar{\lambda}^2 - \delta\bar{\lambda})}{(Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda})(\bar{\lambda}^2 - \delta\bar{\lambda}) + f''(k^*)c^*K\alpha\bar{\lambda}} \\
&= \lim \frac{K\bar{\lambda}(\bar{\lambda} - \delta)}{(Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda})(\bar{\lambda} - \delta) + f''(k^*)c^*K\alpha} = K \lim \frac{\bar{\lambda}}{Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda}} \\
&= -K \lim \frac{\frac{K + \alpha r^{\alpha-1}}{\alpha(1-\alpha)r^{\alpha-2}}}{Kr + r^\alpha + \frac{Kr + \alpha r^\alpha}{1-\alpha}} = -K \lim \frac{K + \alpha r^{\alpha-1}}{\alpha r^{\alpha-1}((2-\alpha)K + r^{\alpha-1})},
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{\alpha \rightarrow 0_+} s(\bar{\lambda}) &= -K \lim_{\alpha \rightarrow 0_+} \frac{K}{\alpha r^{-1}(2K + r^{-1})} = -\infty, \\
\lim_{\alpha \rightarrow 1_-} s(\bar{\lambda}) &= -K.
\end{aligned}$$